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THE DYNAMICS OF AN ELLIPTICAL CRACK IN AN ELASTIC SPACE: SOLUTION USING PADÉ APPROXIMATIONS*

A.V. KAPTSOV and E.I. SHIFRIN

Dynamic problems in the theory of elasticity involving normal cleavage cracks in an unbounded linearly-elastic space under harmonically varying and impact loads are considered. The study involves a reduction of the problem to integrodifferential equations for normal jump displacements on the crack surface. The method used to solve these is based on Padé approximations (PA). The use of this method requires a very accurate representation of the coefficients of a Taylor series expansion of the solution. Thus, the problem of the harmonic effects is solved using PA only for elliptical cracks, when the coefficients of the Taylor series expansion with respect to the wave number are expressed in analytical form. The problem of the impact effect is solved analytically by studying the roots of the Padé approximation for the harmonic problem. The most important characteristics of the solution (stress intensities and the total scattering cross-section) and the effects of the eccentricity of the ellipse on these are investigated. The results obtained are compared with known ones.

1. The pseudodifferential equation for a crack, to the surface of which normal, harmonic stresses are applied has the form /1/

$$p_G\Omega_\beta b(x) = t(x),\ x = (x_1, x_2) \in G;\ b(x) = 0,\ x \notin \bar{G} \tag{1.1}$$
$$\Omega_\beta(\xi) = 2\mu\beta^{-2}[\xi^2(\xi^2-\beta^2)^{1/2} - (\xi^2-\beta^2/2)^2(\xi^2-\alpha^2)^{-1/2}]$$
$$\xi = (\xi_1, \xi_2),\ \xi^2 = |\xi|^2,\ \alpha = \omega/C_d,\ \sigma = \omega/C_s$$

Here, $b(x)$ is the amplitude of the normal jump displacement, $t(x)$ is the amplitude of the forces acting, G is the region in the plane $x_3 = 0$ occupied by the crack, p_G is the compression in the region G, $\bar{G}$ is the closure of the region G, Ω_β is a pseudodifferential operator, μ is the shear modulus, ω is the frequency of variation of the forces applied, C_d and C_s are the longitudinal and transverse propagation velocities (respectively) and the value of $S^{1/2}$ is chosen to be positive for $S > 0$ and $-i|S|^{1/2}$ for $S < 0$.

Eq.(1.1) was obtained on the assumption that the Sommerfeld condition is satisfied at infinity and that in the deformation process overlapping of the crack surfaces does not occur. This may be ensured, for example, by the presence of an initial opening in the crack or an additional static force. We note that by virtue of linearity, Eq.(1.1) also corresponds to the problem of the scattering of plane harmonic elastic waves by a crack. Equations of type (1.1) also arise in the case of the action of elastic loads, after application of a Fourier or Laplace transformation in the time domain.

The problems considered here have been studied numerically in a number of papers. Some of these use methods applied only to specific forms of cracks /2-7/. Eq.(1.1) was also solved in /8/ using a two-basis projection method /9/ applied to the problem of cracks of arbitrary shape.

The idea of the Padé approximation involves the rearrangement of a function represented by a Taylor series with respect to a basis of rational functions, which enables us to broaden its area of convergence and study the behaviour of the function in the complex domain. A detailed description of the properties and use of PA is given in /10/.

Here, we will only assume the main formulae. Suppose that we are given a power series representing the function

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$$f(x) = \sum_{i=0}^{\infty} C_i x^i$$

A Padé approximation of order $[L/M]$ is a rational function of the form

$$[L/M] = \frac{a_0 + a_1 x + \ldots + a_L x^L}{b_0 + b_1 x + \ldots + b_M x^M} \tag{1.2}$$

the Taylor series expansion (about zero) of which agrees with the expansion of $f(x)$ with an accuracy up to a term of order $L + M + 1$.

$$\sum_{i=0}^{\infty} C_i x^i = [L/M] + O(x^{L+M+1}) \tag{1.3}$$

This condition leads to a system of linear equations in the unknown coefficients a_i and b_j of the PA. A number of theorems give the conditions for the Padé approximation $[L_k/M_k]$ to converge to the original function /10/. In the absence of additional information about the meromorphic function subject to approximation using the Padé approximants, it is advisable to use diagonal $[M_k/M_k]$ or paradiagonal $[M_k \pm J/M_k]$, $M_k \to \infty$ sequences. We note that since the Padé approximation approximates a meromorphic function in a broad area using a finite number of expansion coefficients at a single point, the coefficients of the Taylor series should be computed with some accuracy.

2. We will now find a Taylor series with respect to the wave number to solve (1.1). We assume that the amplitude of the forces applied is constant $(t(x) = 1)$ and we shall seek a solution of Eq.(1.1) in the form

$$b(x, \beta) = \sum_{i=0}^{\infty} b_i(x) \beta^i, \quad x \in G, \quad b_i(x) = 0, \quad x \notin \bar{G} \tag{2.1}$$

Suppose that the result of applying the operator Ω_β to the function $b_j(x)$ is expressed in the region of the crack G by the formula

$$\Omega_\beta b_j(x) = \sum_{k=0}^{\infty} \beta^k V_k(b_j) \tag{2.2}$$

Substituting (2.1) and (2.2) into Eq.(1.1), we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \beta^{k+m} V_k(b_m) = 1 \tag{2.3}$$

Comparison of the coefficients of individual powers of β on the left and right sides of Eq.(2.3) leads to a system of equations

$$V_0(b_0) = 1, \quad V_0(b_k) = -\sum_{j=1}^{k} V_j(b_{k-j}) \tag{2.4}$$

Thus, in order to solve problem (1.1) in the form (2.1), we must obtain an explicit expression for the functions b_j and $V_k(b_j)$. This may be done if we assume that the region G has a canonical form.

We return to the problem of a circular crack of radius a. We denote

$$\psi^\gamma(x) = (a^2 - r^2)^\gamma, \; r = |x| < a; \; \psi^\gamma(x) = 0, \; r \geqslant a;$$

$$b_k = \sum_{p=0}^{N(k)} C_{k,p} \psi^{p+1/2}$$

It was shown in /8/ that the result of applying the operator Ω_β to the function ψ^γ may be represented in the form of an infinite power series (2.2) with respect to the wave number, with complex coefficients. In simplified form, the functions $V_j(\psi^\gamma)$ may be expressed in terms of the hypergeometric function

$$V_j(\psi^\gamma) = \frac{\mu \Phi_j a^{2\gamma+j-1} \Gamma(1+\gamma) \Gamma(1/2)}{(2i)^j \Gamma(\gamma + (j+1)/2)} \, {}_2F_1\left(\frac{3-j}{2}, -\gamma + \frac{1-j}{2}; 1; \frac{r^2}{a^2}\right) \tag{2.5}$$

$$\Phi_0 = \eta^2 - 1, \quad \Phi_1 = 0, \quad \Phi_j = \frac{1}{\Gamma(j/2)} \begin{cases} H^R_{(j-2)/2}, & j = 2, 4, 6 \ldots \\ H^I_{(j-3)/2}, & j = 3, 5, 7 \ldots \end{cases}$$

$$H_p{}^R = \eta^{2p}\left[1 - 2\frac{2p+1}{p+1}\eta^2 + \frac{(2p+1)(2p+3)}{(p+1)(p+2)}\eta^4\right] + \frac{2p+1}{(p+1)(p+2)}$$

$$H_p{}^I = \eta^{2p+1}\left[1 - 8\frac{p+1}{2p+3}\eta^2 + 16\frac{(p+1)(p+2)}{(2p+3)(2p+5)}\eta^4\right] + 8\frac{p+1}{(2p+3)(2p+5)}$$

$$\eta^2 = (1 - 2\sigma)/(2(1 - \sigma))$$

where σ is Poisson's ratio and $\Gamma(z)$ is the Gamma function.

For $\gamma = i + 1/2, (i = 0, 1, 2 \ldots)$, this hypergeometric function degenerates into a polynomial, as a result of which we have

$$V_j(\psi^{i+1/2}) = \sum_{m=0}^{f(i,j)} S^j_{i,m} r^{2m} \tag{2.6}$$

$$S^j_{i,m} = \frac{\mu\Phi_j a^{2i+j-2m}\Gamma(3/2 + i)\Gamma(1/2)\Gamma((j-1)/2)}{(2i)^j\Gamma((j-1)/2 - m)\Gamma(i + j/2 + 1 - m)}$$

$$f(i,j) = \begin{cases} N(j) + i \\ N(j) \end{cases}, \quad N(j) = \begin{cases} j/2, & j = 0, 2, 4 \ldots \\ (j-3)/2, & j = 3, 5, 7 \ldots \end{cases}$$

It follows from formulae (2.5) and (2.6) that $V_1(\psi^{p+1/2}) = 0$, whence $b_1 = 0$. The left side of Eq.(2.4) takes the form

$$V_0(b_k) = \sum_{n=0}^{N(k)} Y_n r^{2n}, \quad Y_n = \sum_{m=n}^{N(k)} C_{k,m} S^0_{m,n}, \; k \neq 1 \tag{2.7}$$

Substituting b_j into the right of (2.4) we obtain the expressions

$$-\sum_{j=2}^{k} V_j(b_{k-j}) = \Pi_1(k) + \Pi_2(k) \tag{2.8}$$

$$\Pi_1(k) = -\sum_{n=0}^{N(k)} \chi^{(1)}_{k,n} r^{2n}, \quad k \geqslant 2, k \neq 3; \quad \Pi_1(3) = 0$$

$$\Pi_2(k) = -\sum_{n=0}^{N(k-3)} \chi^{(2)}_{k,n} r^{2n}, \quad k \geqslant 3, k \neq 4; \quad \Pi_2(2) = \Pi_2(4) = 0,$$

$$\chi^{(s)}_{k,n} = \sum_{m=n}^{N(k-1-s)} \sum_{p=\varphi_s(m,n)}^{N(k-1-s)-m} C_{k-1-s-2m,p} S^{2m+3}_{p,n}, \; s = 1, 2$$

$$\varphi_1(m,n) = \begin{cases} 0, & \text{for } m = 0, 1; \; 1 \leqslant n \leqslant N(k), \; m \geqslant n - 1 \\ n - 1 - m, & \text{for } 1 \leqslant n \leqslant N(k), \; m \leqslant n - 1 \end{cases}$$

$$\varphi_2(m,n) = 0$$

From (2.4), (2.7) and (2.8), we obtain a sequence of systems of equations in the unkown coefficients $C_{k,m}$. We note that formula (2.7) enables us to construct a solution of the static problem of a circular crack to the surface of which axially symmetric stresses of the form

$$p = p_0 + p_1 r^2 + \ldots + p_N r^{2N}.$$

are applied.

In order to solve Eq.(1.1) in the form (2.1) in the case where G is an ellipse of the form $x_1{}^2/a_1{}^2 + x_2{}^2/a_2{}^2 \leqslant 1$, it is natural to use a technique analogous to that described above for a circular crack, but replacing the function ψ^γ by φ^γ, where

$$\varphi^\gamma(x) = (1 - x_1{}^2/a_1{}^2 - x_2{}^2/a_2{}^2)^\gamma, \; x \in G; \; \varphi^\gamma(x) = 0, \; x \notin G.$$

Because of the absence of axial symmetry, it is not possible to manage with the functions φ^γ alone and we need to find the result of applying the operator Ω_β to the functions $T^\gamma_{p,q} = x_1{}^p x_2{}^q \varphi^\gamma$. This problem is very tedious. Thus, here, we will only describe the two main stages of its solution.

In the first stage, we compute the function $\Omega_\beta\varphi^\gamma$ which is expressed as a two-dimensional integral over the plane of the Fourier transform variables written in polar coordinates. First we take the integrals with respect to the radius which, using the equation

$$\exp(ir\cos\vartheta) = J_0(r) + 2\sum_{n=0}^{\infty} i^n I(r)\cos(n\vartheta)$$

reduce to the integrals given in /8/. Then, we compute the integrals with respect to the angle, which reduce to tabulated integrals. The final result is expressed as a power series in the wave number of the form (2.2) where the decomposition coefficients $\gamma = i + {}^1/_2$ are given by

$$V_j(\varphi^{i+1/2}) = \sum_{n=0}^{f(i,j)} \sum_{k=0}^{f(i,j)-n} \Phi_{n,k}^{j,i} y_1^{2n} y_2^{2k}, \quad j = 0, 2, 3, \ldots \tag{2.9}$$

$$V_1(\varphi^{i+1/2}) = 0, \quad y_i = x_i/a_i, \quad x \in G$$

$$\Phi_{n,k}^{j,i} = \frac{\mu\Phi_j a_2^{j-1}\Gamma({}^3/_2 + i)\Gamma({}^1/_2)\Gamma((j-1)/2)}{(2i)^j\,\Gamma((j-1)/2 - k - n)\Gamma(i + j/2 + 1 - k - n)} N_{k,n}$$

$$N_{i,n} = \sum_{m=0}^{i} \sum_{k=0}^{n} \frac{\Gamma^2(k + m + {}^1/_2) E_{k+m}((j-1)/2)(-1)^m V_{k+m}}{\Gamma(k + {}^1/_2)\Gamma(m + {}^1/_2)(k + m + n + i)!\,k!\,m!\,(n-k)!\,(i-m)!}$$

$$V_0 = 1, \quad V_k = 2(-1)^k, \quad k = 1, 2, 3$$

$$E_k(\rho) = \sum_{m=0}^{k} \frac{k!}{m!\,(k-m)!}(-1)_2^m F_1(m + {}^1/_2, \rho; k + 1; \chi^2)$$

$$\chi^2 = (a_1^2 - a_2^2)/a_1^2$$

If $a_1 = a_2 = a$, formula (2.9) is transformed into (2.6).

In the second stage, to compute $\Omega_\beta T_{p,q}^{\gamma}$, we use the following formula (which may be proved by mathematical induction):

$$y_1^p y_2^q \varphi^\gamma = \frac{p!\,q!}{(-2)^{p+q}} \sum_{k=0}^{[p/2]} \sum_{m=0}^{[q/2]} \frac{\Gamma(1+\gamma) D_1^{p-2k} D_2^{q-2m} \varphi^{\gamma+p+q-k-m}}{k!\,m!\,(p-2k)!\,(q-2m)!} \tag{2.10}$$

$$D_s = \partial/\partial y_s, \quad s = 1, 2$$

(the square brackets denote the integral part of a number). Since in the given case the amplitude of the applied forces is constant $(t(x) = 1)$ it is sufficient to use the functions $T_{p,q}^{\gamma}$ for even values of p and q only $(p = 2k, q = 2m)$. By virtue of this, we obtain the following formula for these functions only

$$V_j(y_1^{2k} y_2^{2m} \varphi^{i+1/2}) = \sum_{n=0}^{R(j,i,k,m)} \sum_{r=0}^{R(j,i,k,m)-n} Q_{n,r}^{k,m,n,r} y_1^{2n} y_2^{2r} \tag{2.11}$$

$$R(j,i,k,m) = \begin{cases} \varphi(j,i) + k + m, & j = 2, 4, 6, \ldots \\ \varphi(j,i), & j = 3, 5, 7, \ldots \end{cases}$$

$$Q_{n,r}^{u,v,j,i} = \frac{(-1)^{u+v+1}\mu\Phi_j a_2^{j-1}\Gamma({}^3/_2 + i)\Gamma({}^1/_2)\Gamma((j-1)/2)}{(2i)^{j+2u+v}\Gamma(i + 1 + j/2 + u + v - r - n)(2n)!\,(2r)!} \times$$

$$\sum_{p=0}^{u} \sum_{q=0}^{v} \frac{(2n + 2u - 2p)!\,(2r + 2v - 2q)!\,N_{r+v-q,\,n+u-p}}{p!\,q!\,(2u-2p)!\,(2v-2q)!\,\Gamma((j-1)/2 - r - n - u - v + p + q)}$$

As before, the solution of (1.1) is represented using the series (2.1), where, for an elliptically shaped crack, the function $b_i(x)$ is chosen in the form

$$b_k = \sum_{i=0}^{N(k)} \sum_{j=0}^{i} C_{k,\,j+i(i+1)/2} T_{2i-2j,\,2j}^{1/2}$$

Just as in the case of a circular crack, we obtain $V_1(b_k) = 0$, $b_1 = 0$. Substituting b_k into the left-hand side of Eq.(2.4) leads to the expression

$$V_0(b_k) = \sum_{n=0}^{N(k)} \sum_{m=0}^{N(k)-n} y_1^{2n} y_2^{2m} Y_{n,m}^k \tag{2.12}$$

$$Y_{n,m}^k = \sum_{i=n+m}^{k} \sum_{j=0}^{i} C_{k,\,j+i(i+1)/2} Q_{n,m}^{i-j,\,j,\,0,\,0}$$

Substituting b_k into the right-hand side of Eq.(2.4), we obtain

$$-\sum_{j=2}^{k} V_j(b_{k-j}) = \Pi_1^{el}(k) + \Pi_2^{el}(k) \tag{2.13}$$

$$\Pi_1{}^{el}(k) = \sum_{n=0}^{N(k)} \sum_{m=0}^{N(k)-n} y_1^{2n} y_2^{2m} X_{n,m}^{(1),k}, \quad k \geqslant 2, k \neq 3; \Pi_1{}^{el}(3) = 0$$

$$\Pi_2{}^{el}(k) = \sum_{n=0}^{N(k-3)} \sum_{m=0}^{N(k-3)-n} y_1^{2n} y_2^{2m} X_{n,m}^{(2),k}, \quad k \geqslant 3, k \neq 4; \Pi_2{}^{el}(2) = \Pi_2{}^{el}(4) = 0$$

$$X_{n,m}^{(s),k} = \sum_{w=\tau_s(n,m)}^{N(k-1-s)} \sum_{i=0}^{w} \sum_{j=0}^{i} Q_{n,m}^{i-j,\, j,\, 2w+1+s-2i,\, 0} C_{k-2w+2i-1-s,\, j+i(i+1)/2}$$

$$\tau_1(n,m) = \begin{cases} 0, & n+m < 1 \\ n+m-1, & n+m \geqslant 1 \end{cases}, \quad \tau_2(n,m) = n+m$$

From (2.4), (2.12) and (2.13), we find a sequence of systems of linear equations in the coefficients $C_{k,m}$. We note that formula (2.13) enables us to construct a solution of the static problem of an elliptical crack for a load given in terms of a polynomial containing only terms of even degree in each variable. Similarly, we may obtain formulae to solve the problem for loads in the form of an arbitrary polynomial.

3. We will consider the results of calculations for the problem of elliptical cracks, involving PA solutions written in the form of power series as in Sect.2. In what follows, we took $\sigma = 0.25$. We note that use of the Taylor series gives a solution only in a small neighbourhood of $\beta = 0$ (static problem), since the function which determines the dependence of the solution on the wave number has poles.

Fig.1 shows curves of $\lambda = |N|/N_{st}$ as a function of the given wave number at the points of the ellipse $(0, a_2)$ (the continuous lines) and $(a_1, 0)$ (the dashed lines) for ellipses with axis ratios 1 : 1, 1 : 2, 1 : 4 and 1 : 8 (curves *1, 2, 3, 4*). Here $|N|$ is the modulus of the amplitude of the stress intensity and N_{st} is the stress intensity at the same point for the static problem with a uniform load of a single intensity. In the calculations, we considered PA of the form $[L/L]$, $[L+1/L]$, $[L/L+1]$ up to $L = 9$. The restriction of the order of the PA is associated with the loss of accuracy in computing the coefficients of the Taylor series expansion of the solution according to the formulae of Sect.2. All the calculations were carried out with double precision and the results for the PA $[9,9]$ are given.

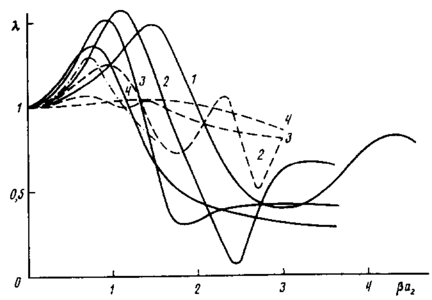

Fig.1

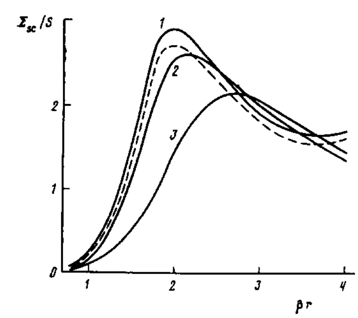

Fig.2

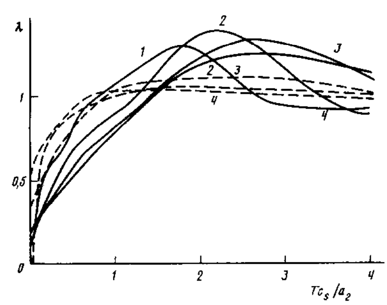

Fig.3

To illustrate how quickly the results of the calculations are determined, depending on the value of the wave number, and to determine the sensitivity of the calculated values to the order of the PA $[L/L]$, we list below the dependence (on the order of the PA) of the first maximum $\lambda^{(1)}$, the first minimum $\lambda^{(2)}$ and the second maximum $\lambda^{(3)}$ $(\lambda^{(\varkappa)} = |N^{(\varkappa)}|/N_{st}, \varkappa = 1, 2, 3)$

together with the corresponding resonance values of the wave number $\beta^{(\varkappa)}a$ corresponding to the problem of a circular crack

$[L/L]$	[3/3]	[4/4]	[5/5]	[6/6]	[7/7]	[8/8]	[9/9]
$\lambda^{(1)}$	1.4701	1.4788	1.4791	1.4791	1.4791	1.4791	1 4791
$\beta^{(1)}a$	1.5	1.5	1.5	1.5	1.5	1.5	1.5
$\lambda^{(2)}$	0.3274	0.4071	0.3971	0.4023	0.3980	0.3975	0.3975
$\beta^{(2)}a$	5.2	3.0	3.0	3.0	3.0	3.0	3.0
$\lambda^{(3)}$	—	0.6422	0.7570	0.6929	0.7876	0.8296	0.8358
$\beta^{(3)}a$	—	5.2	5.5	4.3	4.3	4.3	4.3

It is clear from Fig.1 that for elliptical cracks, as the eccentricity of the ellipse increases, the value of the first maximum $\lambda^{(1)}$ at the point $(0, a_2)$ and the corresponding resonance value $\beta_{\text{res}}a_2$ tend to values corresponding to solutions of the planar problem /11/ (the dashed curve). We note that the value of $\lambda^{(1)}$ at the point $(0, a_2)$ depends non-monotonically on the eccentricity of the ellipse. Increasing the eccentricity of the ellipse also leads to degraded convergence of the PA since in the given range of variations of $[L/L]$ for the ellipse 1 : 8 at the point $(0, a_2)$ only the first maximum is stabilized. This is clearly due firstly to purely computational reasons and secondly to the fact that in the planar problem the decomposition of the jump displacements in an asymptotic series contains not only powers but also logarithmic terms /11/.

Comparison of the stress intensities derived using the method based on PA and the two-basis projection method /8, 9/, showed that they agree to within 5% for wave numbers in the given band.

One important characteristic which determines the effectiveness of the scattering of planar elastic waves by a defect is the total scattering cross-section Σ_{sc} which is expressed in terms of the ratio of the flux density of the energy of the scattered waves averaged over the period of the oscillations and integrated over all directions to the flux density of the incident waves averaged over the period of oscillation /12/. In the case of normal incidence of planar longitudinal waves on the crack, the total scattering cross-section is given by the following formula from /12/

$$\Sigma_{sc} = \beta/\eta \iint_G \operatorname{Im}(b)\, dx$$

Fig.2 shows a graph of $\Sigma_{sc}(\beta r)/S$ for cracks of various shapes, where S is the area of the region of the crack and r is the radius of a circle of area S. It illustrates the cases of ellipses with axis ratios 1 : 1, 1 : 2 and 1 : 4 and a square (continuous curves *1, 2, 3* and the dashed curve, respectively). Calculations were carried for all the cracks using the two-basis projection method /8, 9/ and for the elliptical cracks using the method based on PA. Since the value of Σ_{sc} as $\beta \to 0$ is of the order of β^4, PA were constructed for the function Σ_{sc}/β^4. Moreover, the Taylor series expansion of $\operatorname{Im}(b)$ in terms of β contains only odd powers, whence we took β^2 as an independent variable for this function. Thus, the order of the PA used was lower than in the investigation of the stress intensities. The figure shows the results corresponding to PA of order [3/4]. The results of the calculations obtained by the two methods agree. They also agree with previous results for the problem of a circular crack /13/.

Based on numerical calculations, we may assume that the following isoperimetric inequality is satisfied: for all cracks of the same area the greatest value of Σ_{sc} with respect to β is a maximum for circular cracks.

4. We consider the non-stationary problem of the effect of shocks normal to the plane of the crack for a typical case of a semi-infinite step load $H(\tau)$ of unit height, applied at time $\tau = 0$ ($H(\tau)$ is the Heaviside function). Application of a Laplace transformation in the time domain with parameter s to the equations of the theory of elasticity leads to equations which are identical with the equations for the problem of determining fluctuations subject to the substitution $\omega^2 = -s^2$ /5, 7/. Consequently, the pseudodifferential equation of the non-stationary problem has the form

$$p_G \Omega_s^* u^* = \frac{1}{s}, \quad x \in G, \quad u^* = 0, \quad x \notin \bar{G} \tag{4.1}$$

$$u^*(x, s) = \int_0^\infty e^{(-s\tau)} u(x, \tau)\, d\tau, \quad \frac{1}{s} = H(s^*)$$

where the pseudodifferential operator symbol is derived from the symbol Ω_β by the substitution $\omega = is$. From (1.1) and (4.1), its follows that $u^*(x, s) = b(x, is)/s$. Thus, the Laplace expansion of the solution of the impact problem in a series with respect to the transformation parameter is expressed in terms of the Taylor series expansion with respect to the wave number in the problem of determining the oscillations. Inversion of PA of order $[L/M]$ approximating the solution of $u^*(x, s)$ does not appear to be difficult, since $[L/M]$ is a rational function /14/.

In particular, if all the M roots of the replacement of the approximation of order $[L/M]$ are simple, then

$$u^*(x, s) = \frac{K_0}{s} + \sum_{j=1}^{M} \frac{K_j}{s - s_j}, \quad u(x, \tau) = K_0 + \sum_{j=1}^{M} K_j e^{(s_j \tau)} \tag{4.2}$$

$$K_j = \lim_{s \to s_j} u^*(x, s)(s - s_j)$$

where K_j and s_j are complex numbers.

Fig.3 shows the case of PA of order $[9, 9]$ including curves of the variation of the stress intensities as a function of time at the points of the ellipse $(0, a_2)$ and $(a_1, 0)$, where the notation for the curves is as in Fig.1. As the eccentricity of the ellipse increases, the solutions at the point $(0, a_2)$ converge to a solution of the planar problem /15/ and in the case of a circle, they compare well with well-known results /5/ and with solutions obtained by the authors using the two-basis projection method. It is clear that for $\tau = 0$, we have $N(0) \neq 0$, i.e. there is a computational error (which decreases as the order of the PA increases).

Below, for the example of a circular crack, we show the process of stabilizing the poles $s_j = a_j + ib_j$ of PA of the form $[L/L]$ as their order increases, here $a_1 = -0.595$ and $b_1 = 1.639$ for the given values of $[L/L]$.

$[L/L]$	[4/4]	[5/5]	[6/6]	[7/7]	[8/8]	[9/9]
a_2	—2.332	—2.239	—0.989	—0.708	—0.771	—0.713
b_2	4.249	4.713	4.006	4.249	4,359	4.351
a_3	—	—	—3.081	—4,293	—3.904	—3,850
b_3	—	—	2.851	2.581	2,440	2.308

We note that the imaginary parts of the first two poles correspond approximately to the first and second maxima of the frequency curve (see Fig.1, the continuous curve *1*). As the eccentricity of the ellipse increases, the convergence of the poles of the corresponding PA deteriorates. For example, if we use PA of the form $[L/L]$ up to $L = 9$, in the case of a circular crack we find three poles close to zero, which for the ellipse 1 : 8 and the same range of PA are the only roots close to zero.

5. In conclusion, we list the individual features of the method of solution employed here.

1. As far as the requirements on the accuracy of computation of the coefficients of the Taylor series expansion of the solution as concerned, the set of solvable problems is limited only by the shape of the cracks for which we may obtain an analytical expression for the terms of the series. However, the fact that we have been successful in the case of elliptical cracks suggests that this set of problems is quite large.

2. Unlike previous methods where, in problems to determine oscillations, it was necessary to find a numerical solution for each value of the wave number, here, we obtain analytical expressions which approximate solutions directly in some region of variation of the wave numbers.

3. The solution of non-stationary problems usually involves difficult and insufficiently accurate procedures for numerical inversion of Laplace or Fourier transformations. In the method based on Padé approximation, this solution is obtained by analytical inversion of an approximate expression.

4. In this method, the computation time depends mainly on the calculation of the Taylor series coefficients. Computation of 18 terms required less than 10 minutes in the case of a circle and around 6 hours for an ellipse on an EC 1055 or an EC 1055 M computer. All the previous calculations both to determine the oscillations and the impact effects take seconds.

5. It is possible, in principle, to increase the accuracy of the arithmetic operations using a large number of Taylor series terms and thus determine an approximate solution in a

range of variation of the wave numbers as large as desired.

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INVARIANT SOLUTIONS OF THE EQUATIONS OF THE NON-ISOTHERMAL STATIONARY FLOW OF A VISCOUS FLUID IN TUBES*

R.N. BAKHTIZIN and R.K. MUKHAMEDSHIN

The group properties /1/ of a system of equations describing flows in tubes of fluids the viscosity of which depends on the temperature are investigated for large Peclet numbers. It is shown that for exponential and power dependences there is an extension of the main group of transformations. For these cases, invariant solutions which have a physical meaning are considered.

The equations describing the motion of a viscous fluid in a cylindrical tube may be written, in dimensionless form as follows for $\delta \ll 1, \mathrm{Pe} \gg 1$ /2/:

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